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MECHANICS

Determination of the Mass-Flow Rate of Blood in a Blood Vessel Using Natural Frequencies of Flexural Vibrations

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Abstract—The natural flexural vibrations of a blood vessel with moving blood are investigated. The vessel is subjected to the action of tensile force and pressure in it. The forces of inertia of the blood vessel, as well as the Coriolis and centrifugal forces induced by the motion of blood, are taken into account. The wave numbers are determined, and the frequency equation is found using the boundary conditions. Two frequencies of flexural vibrations can determine the speed parameter, the relative mass of blood per unit length of the blood vessel, and, as a consequence, the mass-flow rate of blood through the blood vessel. The results obtained can be used for the acoustic method of determining the blood speed, the relative blood mass per unit length of the blood vessel, and the mass-flow rate of blood through it.

Keywords: blood vessel, flexural vibrations, natural frequencies, velocity, direct and inverse problems

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In [1], we showed after the statistical processing of the data obtained that there is a dependence of the diameter of the total femoral vein and the blood flow in it on the level of intra-abdominal pressure. The investigation was carried out with participation of ten volunteers for which we measured the diameter of the femoral vein and the speed of blood flow in it during laparoscopy using an ultrasound at different levels of pressure in the abdomen generated by an insufflator. In view of the presence of the proportional almost linear functional dependence of the diameter of the femoral vein on the level of intra-abdominal pressure, we proposed to use this indicator for monitoring the dynamics of intra-abdominal pressure. As was noted in [2], there is a gradual increase in the diameter of the lumen of the central arteries with age, along with a thickening and compaction of the vascular wall.

We consider the natural frequencies of flexural vibrations of a blood vessel with blood under pressure. The case of hard pinching of edges of the blood vessel is investigated. In the extensive literature on pipeline vibrations, the conditions of "articulated support" and "free ends" also widely used; however, as noted in [3], they are not obvious, need to be substantiated, and possibly require replacement by others. The inverse problem of determining the speed parameter and the relative mass of blood per unit length of a blood vessel is solved.

PROBLEM FORMULATION

We investigate the natural frequencies of flexural vibrations of a blood vessel of length L, which contains blood moving at a constant velocity V. The blood vessel is an elastic medium under pressure and tensile force T. It is required to determine the speed parameter and the relative mass of blood per unit length of the blood vessel from the natural frequencies of flexural vibrations.

The equation of flexural vibrations of the blood vessel according to the Kirchhoff model has the form [4–7]

$$EJ\frac{\partial^4 w_*}{\partial x^4} + (\rho_i F_i V^2 + P_i F_i - T)\frac{\partial^2 w_*}{\partial x^2} + 2\rho_i F_i V\frac{\partial^2 w_*}{\partial x \partial t} + [\rho F + \rho_i F_i]\frac{\partial^2 w_*}{\partial t^2} + q_s w_* = 0, \tag{1}$$

where E, ρ, J, and F are the elasticity modulus, the density, the axial moment of inertia, and the cross-section area of the blood vessel; ρ_i, P_i, and F_i are the density, the pressure inside the blood vessel, and the cross-section area of the blood vessel lumen; w_* is the blood vessel deflection, q_s is the environmental stiffness coefficient, x is the coordinate directed along the

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blood vessel axis, and t is the time. Passing to dimensionless quantities,

$$\xi = \frac{x}{L}, \quad w = \frac{w_*}{L}, \quad \tau = \nu t, \quad \nu^2 = \frac{EJ}{\rho F L^4},$$

$$R = \frac{(T - P_i F_i)L^2}{EJ}, \quad \alpha = \frac{V}{\nu L}, \quad \beta = \frac{\rho_i F_i}{\rho F}, \quad \gamma = \frac{q_s L^4}{EJ},$$

$$p = \alpha^2\beta - R, \quad q = 2\alpha\beta\Omega, \quad r = -(1+\beta)\Omega^2 + \gamma,$$

$$\Omega^2 = \frac{\rho F L^4 \omega^2}{EJ}, \quad F_i = \pi R_i^2,$$

$$F = \pi[(R_i + h)^2 - R_i^2], \quad J = \frac{\pi[(R_i + h)^4 - R_i^4]}{4},$$

we make the substitution $w = W(\xi)\exp(i\Omega\tau)$ and obtain the equation that determines the shape of the flexural vibrations of the blood vessel

$$\frac{\partial^4 W}{\partial \xi^4} + p\frac{\partial^2 W}{\partial \xi^2} + iq\frac{\partial W}{\partial \xi} + rW = 0, \qquad (2)$$

where ω is the circular frequency, R_i is the inner radius of blood vessel, α is the speed parameter, and β is the parameter of blood mass in the blood vessel.

The boundary conditions for the blood vessel pinched at the edges are

$$w = 0, \quad \frac{\partial w}{\partial \xi} = 0 \quad (\xi = 0, 1). \qquad (3)$$

We determine the general solution of Eq. (2) in the form of $W(\xi) = \exp(k\xi)$; then, we obtain the characteristic equation for finding the unknown values of the complex wave parameter $k_j = k_j(\alpha, \beta, R, \Omega)$, $j = 1, 2, 3, 4$:

$$k^4 + pk^2 + iqk + r = 0. \qquad (4)$$

From the Ferrari formula, we find the wave numbers $k_j = k_j(\alpha, \beta, R, \Omega)$ and write the general solution of Eq. (2) in the form

$$W(\xi) = \sum_{j=1}^{4} C_j \exp(k_j\xi). \qquad (5)$$

Substituting Eq. (5) into boundary conditions (3), we obtain a homogeneous set of linear equations with respect to the unknown constants C_j. In order that the constants C_j would not be zero simultaneously, it is necessary that the determinant of the main matrix be zero. This condition gives the frequency equation [8]

$$D = \begin{vmatrix} 1 & 1 & 1 & 1 \\ k_1 & k_2 & k_3 & k_4 \\ e^{k_1} & e^{k_2} & e^{k_3} & e^{k_4} \\ k_1 e^{k_1} & k_2 e^{k_2} & k_3 e^{k_3} & k_4 e^{k_4} \end{vmatrix} = 0. \qquad (6)$$

In [3], the theorem was proved that, for any α and β ($R = 0$), the eigenfrequencies Ω of boundary-value problem (1), (3) are real.

Thus, in the simplest model of a blood vessel with moving blood under pressure, the parameters R, α, and β appear, which depend on the tension force T in the blood vessel, the pressure P_i inside the blood vessel, the lumen cross-section area F_i of the blood vessel, and the speed V of the blood flow inside the blood vessel. We dwell on the influence of these factors on the natural frequencies of flexural vibrations. The dependences of the first and second natural frequencies of the flexural vibrations of the rod on the speed parameter α for various values of the parameter R and the analysis of the results are presented in [8]. In contrast to [8], we give here the formulation and solution of the inverse problem of determining the speed parameter and the relative linear mass of moving blood in the blood vessel from two natural frequencies of flexural vibrations. The dimensionless mass flow rate m of the blood vessel is determined by the formula

$$m = \alpha\beta = \frac{\rho_i F_i V}{\rho F \nu L} = \frac{\rho_i F_i V L}{\sqrt{\rho F E J}}.$$

For $\alpha = 0$ and $R = 0$, the eigenfunctions $W(\xi)$ are real and coincide with the eigenfunctions of a rod with rigidly pinched ends [8].

If $R = 0$, $\beta = 1$, and $\gamma = 0$, then the characteristic equation (4) admits the factorization, and its roots k_j are explicit functions of the frequency Ω. For the real eigenfrequencies $\Omega_n(\alpha)$, a reasonably simple secular equation is obtained, which is solved by one of the numerical methods [9, 10].

In other cases, the roots of the characteristic equation are found using the Ferrari formulas. These formulas, according to the authors of [8], were hardly used in problems of mathematical physics and mechanics (a rare exception is [8, 10]).

DIRECT PROBLEM

The calculations were carried out for the following parameters of the blood vessel: the elastic modulus of the vein [11] $E = 0.853 \times 10^6$ N/m^2, the density $\rho = 1058.2$ kg/m^3, the wall thickness of the blood vessel $h = 0.5$ mm, the inner radius of the blood vessel $R_i = 6$ mm, the axial force $T = 0$, the pressure inside the blood vessel $P_i = 10$ mm Hg = 1333.22 Pa, the blood density in the blood vessel $\rho_i = 1052$ kg/m^3, the blood speed inside the blood vessel $V = 0.1$ m/s, the length of the blood vessel between the "supports" $L = 0.1$ m, and the parameters $R = -4.602$, $\alpha = 0.0796$, and $\beta = 5.72$. The solution of the direct problem for the vessel with these parameters gives that the lower two eigenfrequencies of the blood vessel are $f_1 = 16.21$ Hz and $f_2 = 46.17$ Hz ($\Omega_1 = 8.114$ and $\Omega_2 = 23.10$).

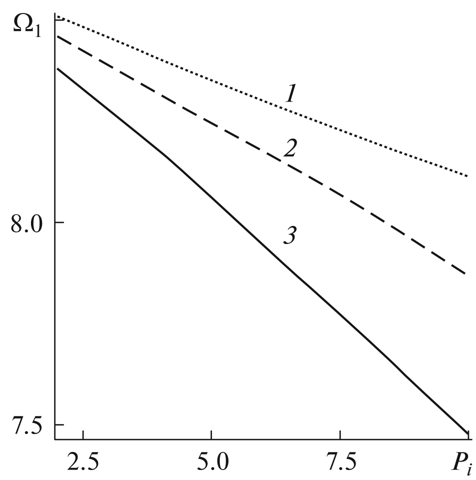

Fig. 1. Dependence of the first natural frequency of the flexural vibrations of the blood vessel on the pressure P_i (mm Hg) inside it for the parameter $\beta = 5.72$ and various values of the elastic modulus $E = 0.853$, 0.6, and 0.4 MPa (curves *1–3*, respectively).

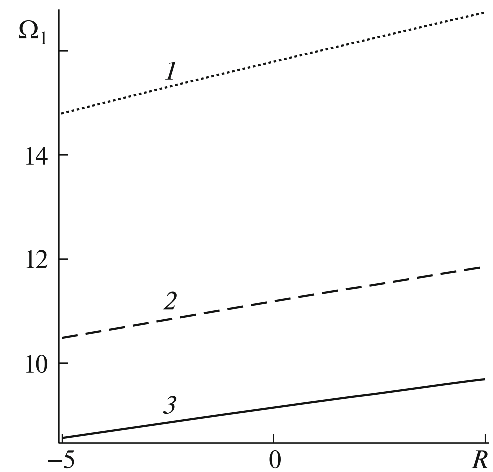

Fig. 2. Dependence of the first natural frequency of the flexural vibrations of the blood vessel on the parameter R for the speed parameter $\alpha = 0.0796$ and various values of the parameter $\beta = 1$, 3, and 5 (curves *1–3*, respectively).

In Fig. 1, we show the dependence of the first natural frequency of the flexural vibrations of the blood vessel on the pressure inside the blood vessel. With increasing pressure inside the blood vessel and decreasing elastic modulus, the frequencies of the flexural vibrations decrease.

In Fig. 2, we show the dependence of the first eigenfrequency of flexural vibrations of the blood vessel on the parameter R. With increasing parameter R, the frequencies of flexural vibrations increase. In Fig. 3, we show the dependence of the first natural frequency of the flexural vibrations of the blood vessel on the parameter β. It can be seen that, with increasing parameter β, the frequencies of flexural vibrations decrease.

INVERSE PROBLEM

At the point $M_0(\alpha_0, \beta_0)$, $D_1(\alpha_0, \beta_0, \Omega_1) = u_1$, and $D_2(\alpha_0, \beta_0, \Omega_1) = u_2$, therefore, we can write

$$\frac{\partial D_1}{\partial \alpha} d\alpha + \frac{\partial D_1}{\partial \beta} d\beta = -u_1,$$
$$\frac{\partial D_2}{\partial \alpha} d\alpha + \frac{\partial D_2}{\partial \beta} d\beta = -u_2.$$

From these formulas, we determine $d\alpha$, $d\beta$, and further, $\alpha_0 = \alpha_0 + d\alpha$, $\beta_0 = \beta_0 + d\beta$. The process of successive approximations continues until the condition of accuracy is satisfied. The solution to this set of equations is determined by the method of successive approximations in the region of the unambiguous dependence of the parameters α and β on the frequency of flexural vibrations of the blood vessel. The solution of the inverse problem for the blood vessel with the above data for $\gamma = 0$, $R = 0$, $\Omega_1 = 9.1$, and $\Omega_2 = 25.1$ ($f_1 = 18.18$ Hz and $f_2 = 50.16$ Hz) gives that $\alpha = 0.0978$, $\beta = 5.034$ ($V = 1.543$ m/s, $\rho_i = 924.8$ kg/m^3). In Fig. 4, we show the dependence on the first frequency Ω_1 of flexural vibrations of the mass-flow rate m through the blood vessel. From the two frequencies of flexural vibrations, it is possible to determine the speed parameter α, the parameter β, or the mass of fluid per unit length and the dimensionless mass-flow rate m of fluid in the blood vessel.

It was found that, with an increase in the force parameter, the frequencies of the flexural vibrations increase. It is shown that, with an increase in the mass

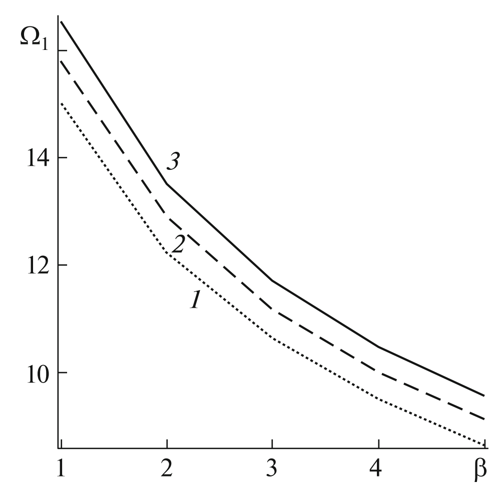

Fig. 3. Dependence of the first natural frequency of the flexural vibrations of the blood vessel on the parameter β for the speed parameter $\alpha = 0.0796$ and various values of the parameter $R = -4$, 0, and 4 (curves *1–3*, respectively).

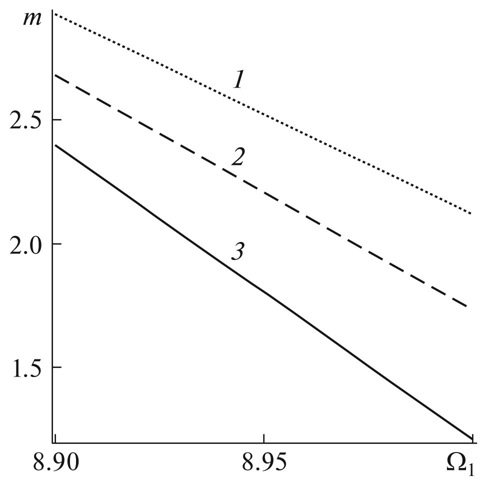

Fig. 4. Dependence of the mass flow m on the first frequency Ω_1 of flexural vibrations through the blood vessel for different frequencies Ω_2 of flexural vibrations: curve *1,* 25.1; *2,* 25.0; and *3,* 24.9.

of the blood vessel with liquid per unit length, a decrease in the natural frequencies of the flexural vibrations of the vessel occurs. From two frequencies of flexural vibrations, we can determine the speed parameter, the mass of blood per unit length, and the dimensionless mass flow of blood through the blood vessel.

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