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Lyapunov Quantities for Andronov–Hopf Bifurcation Problem in Reaction–Diffusion Systems

G. R. Abushahmina[1*], N. I. Gusarova[2], and M. G. Yumagulov[3***]**

(Submitted by T. K. Yuldashev)

[1] *Bashkir State Medical University, Ufa, 45000 Russia*

[2] *Rybinsk State Aviation Technological Academy, Rybinsk, 152934 Russia*

[3] *Bashkir State University, Ufa, 450074 Russia*

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Abstract—The problem of the first Lyapunov quantity constructing on the Andronov–Hopf bifurcation problem in infinite-dimensional dynamical systems is considered. A general scheme of obtaining new formulas for the Lyapunov quantity in terms of the original equations is proposed. The "reaction–diffusion" equation in a limited region and in a situation when there are no flows of reacting components across the boundary of the region is considered as the main object of research. For this equation, the Andronov–Hopf bifurcation conditions in the vicinity of a spatially homogeneous equilibrium point are obtained, necessary conditions for the stability of emerging solutions are specified. New formulas for the first Lyapunov quantities and transcriticity indices of the problem, leading to algorithms for constructing these quantities are proposed. The specifics of these formulas are indicated in the situation when the nonlinearity begins with cubic terms. The proposed formulas make it possible not only to efficiently calculate the Lyapunov quantities, but also to conduct a study of the properties of bifurcations in reaction–diffusion systems under new conditions.

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1. PROBLEM STATEMENT

A key role in the bifurcation theory of dynamical systems and its applications is played by so-called Lyapunov quantities allowing one to determine important properties of bifurcations like the stability of emerging solutions, the direction of bifurcations, the stability properties of a weak focus, etc. There is a series of approaches allowing one to calculate the Lyapunov quantities in problems with various bifurcation scenarios (see, for instance, [1–5] and the references therein). These approaches are aimed at studying bifurcations in finite-dimensional dynamical systems. When passing to the problems of bifurcations in infinite-dimensional dynamical systems, a preliminary substantial transformation of the initial equations based on the theorem of central manifold is required [6].

Proposed in [7] operator approach allows calculating Lyapunov quantities in problems on the main bifurcation scenarios for finite-dimensional dynamical systems. This approach allows one to obtain new formulas for Lyapunov quantities in terms of the original equations. This article is devoted to the development of this approach for the Andronov–Hopf bifurcation problem in infinite-dimensional dynamical systems. It offers a general scheme that allows one to obtain new formulas for Lyapunov quantities in this problem in terms of the original equations. Proposed formulas allow not only to

[*] E-mail: abushahmina_g@mail.ru

[**] E-mail: gusarova-73@mail.ru

[***] E-mail: yum_mg@mail.ru

calculate Lyapunov quantities efficiently, but also to conduct a study of the properties of bifurcation under new conditions.

The "reaction–diffusion" equation is considered as the main object of research. The systems leading to such an equation find numerous applications in chemistry, biology, geology, physics, ecology, etc. As a rule, such systems depend on various parameters, that naturally leads to questions about stability and bifurcations. The general literature is devoted to the study of corresponding problems (see, for instance, [6, 8–16]).

We consider the "reaction–diffusion" system described by the differential equation

$$\frac{dw}{dt} = A(\mu)w + K(\mu)\Delta w + h(w,\mu), \tag{1}$$

where $w = w(x,y) = \begin{bmatrix} u(x,y) \\ v(x,y) \end{bmatrix}$, $A(\mu) = [a_{ij}(\mu)]$, $K(\mu) = [k_{ij}(\mu)]$ square real matrices of order 2, smoothly depending on the scalar parameter μ, in this case, $k_{ij}(\mu) > 0$; $K(\mu)$ is the diffusion matrix; nonlinearity $h(w,\mu)$ begins with quadratic in w terms, it is represented as

$$h(w,\mu) = h_2(w,\mu) + h_3(w,\mu) + ... \tag{2}$$

in which $h_2(w,\mu)$ involves quadratic in w terms, $h_3(w,\mu)$ involves cubic terms etc. Finally, Δ is the Laplace operator: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

Equation (1) is studied on a rectangle

$$\Omega = \{(x,y) : 0 \leqslant x \leqslant 2\pi, \quad 0 \leqslant y \leqslant 2\pi\} \tag{3}$$

with Neumann boundary conditions

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0; \tag{4}$$

where $\partial\Omega$ is rectangle Ω border. The solution of the problem (1), (4) is called a function $w(x,y;t)$ that satisfies equation (1) and boundary conditions (4) for all t and $(x,y) \in \Omega$.

By $L_2(\Omega)$ we denote common Hilbert space of vector functions $w(x,y)$ defined in Ω. Further, by $C(\Omega)$ and $C^2(\Omega)$ we denote respectively the space of continuous and the space of doubly continuously differentiable functions on Ω. For $w(x,y) \in C^2(\Omega)$ we denote norm

$$||w||_2 = \left(\int\limits_{\Omega} \sum_{|\alpha| \leqslant 2} ||D^\alpha w||^2 dxdy \right)^{1/2}, \tag{5}$$

where D^α is differentiation operator: $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}$, $|\alpha| = \alpha_1 + \alpha_2$; $||\cdot||$ is the Euclidean norm in R^2. Finally by $W_2^2(\Omega)$ we denote the Sobolev space, which is a completion of the space $C^2(\Omega)$ by norm (5). We also define the set

$$C_0^2(\Omega) = \left\{ w \in C^2 : \left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = 0 \right\}. \tag{6}$$

According to the standard theory (see, for instance, [18]) the Laplace operator $\Delta : C^2 \to C$ can be extended to a closed self-adjoint operator $\Delta_0 : L_2 \to L_2$ with the domain G, formed by the closure in W_2^2 of set (6). The domain G of operator Δ_0 becomes Banach space if the norm (5) is introduced in it. In this Banach space the operator Δ_0 is bounded. Accordingly, the linear operator $\Delta_0 : W_2^2 \to W_2^2$ with the domain G is bounded. The equation (1) has a zero solution $w \equiv 0$. In this paper we study the problem of the Andronov–Hopf bifurcation in the neighborhood of this solution. The main attention is paid to finding sufficient conditions for this bifurcation, as well as to the construction of corresponding Lyapunov quantities.

2. THE ANDRONOV–HOPF BIFURCATION SIGNS

The Andronov–Hopf bifurcation means the occurrence of non-stationary periodic solutions $\omega(x, y, t, \mu)$ of small amplitude of the equation (1) when parameter μ passes through some critical quantity μ_0 in the the neighborhood of the point $w = 0$.

Bifurcating solutions $\omega(x, y, t, \mu)$ of equation (1) at small $|\mu - \mu_0|$ arise usually in one of three cases: $(S1)$ $\mu > \mu_0$; $(S2)$ $\mu < \mu_0$; $(S3)$ $\mu = \mu_0$. The latter case is called degenerate; it is typical, for example, for linear systems. The first two cases occur when a certain condition of non-degeneracy with respect to nonlinearity (2) is met. If this condition is met in the cases of $(S1)$ and $(S2)$ each μ corresponds to exactly one non-zero periodic solution $\omega(x, y, t, \mu)$ of small amplitude, and the function $\omega(x, y, t, \mu)$ smoothly depends on μ and there is a relation: $\max\limits_{t,(x,y)\in\Omega} ||\omega(x, y, t, \mu)|| \to 0$ when $\mu \to \mu_0$. Finally, the period $T(\mu)$ of solutions $\omega(x, y, t, \mu)$ also smoothly depends on μ and there is a relation: $T(\mu) \to T_0$ when $\mu \to \mu_0$; the quantity of T_0 will be discussed below.

We present statements regarding sufficient conditions under which the Andronov–Hopf bifurcation takes place. Let's define the functions

$$\varphi(\mu) = \mathrm{tr}A(\mu), \quad \psi(\mu) = \det A(\mu). \tag{7}$$

Theorem 1. *Let $\varphi(\mu_0) = 0$, $\varphi'(\mu_0) \neq 0$ and $\psi(\mu_0) > 0$. Then μ_0 is the Andronov–Hopf bifurcation point of the problem* (1), (4).

Let us now give a more general statement. For this purpose, for integers m, n we define the matrices

$$B_{mn}(\mu) = A(\mu) - (m^2 + n^2)K(\mu) \tag{8}$$

and functions

$$\varphi_{mn}(\mu) = \mathrm{tr}B_{mn}(\mu), \quad \psi_{mn}(\mu) = \det B_{mn}(\mu). \tag{9}$$

Theorem 2. *Let for some integers m_0, n_0 the relations $\varphi_{m_0n_0}(\mu_0) = 0$, $\varphi'_{m_0n_0}(\mu_0) \neq 0$ and $\psi_{m_0n_0}(\mu_0) > 0$ hold. Then μ_0 is the Andronov–Hopf bifurcation point of the problem (1), (4).*

Obviously, when $m_0 = n_0 = 0$, Theorem 2 turns into Theorem 1.

2.1. Proof of Theorem 2

Note first that system (1) generates (see, for instance, [6]) a smooth local half-stream $F_t(w)$ on G. The differential $D_wF_t(0)$ is a compact semigroup with an infinitesimal generator

$$S(\mu) = A(\mu) + K(\mu)\Delta. \tag{10}$$

Next, we give the necessary condition that the quantity of μ_0 be the Andronov–Hopf bifurcation point of the problem (1), (4). Such is the requirement that the linear bounded operator $S(\mu_0) : W_2^2 \to W_2^2$ (with the domain G) had a pair of purely imaginary eigenvalues.

In this regard, we note that the next statement is true.

Lemma 1. *The set of eigenvalues of the operator $S(\mu) : W_2^2 \to W_2^2$ coincides with the set of all eigenvalues of all matrices (8).*

In other words, each eigenvalue λ of the operator $S(\mu)$ is an eigenvalue of the matrix (8) for some integers m, n. Conversely, the eigenvalues of all matrices (8) are the eigenvalues of the operator $S(\mu)$.

The proof of Lemma 1 is based on the fact that, first, the eigenfunctions of the operator $\Delta_0 : W_2^2 \to W_2^2$ (with the domain G) are nonzero vector functions of the form

$$w(x, y) = \begin{bmatrix} a \\ b \end{bmatrix} \cos(mx)\cos(ny) \quad (a, b \text{are real numbers}), \tag{11}$$

which correspond to the eigenvalues $\lambda_{mn} = -m^2 - n^2$. Second, the non-zero function (11) is also the eigenfunction for operator $S(\mu) : W_2^2 \to W_2^2$, if the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ is the eigenvector of the matrix (8).

Due to Lemma 1, the necessary condition for the Andronov–Hopf bifurcation of the problem (1), (4) hold if, for some m and n, the matrix (8) for $\mu = \mu_0$ has a pair of purely imaginary eigenvalues. And this fact follows from the assumption that $\varphi_{m_0n_0}(\mu_0) = 0$ and $\psi_{m_0n_0}(\mu_0) > 0$. As for the sufficient bifurcation condition, it is a consequence of the assumption $\varphi'_{m_0n_0}(\mu_0) \neq 0$. The Theorem 2 is proved.

3. ON THE STABILITY OF BIFURCATION SOLUTIONS

Let the assumptions of Theorem 2 hold. Then the bifurcating solutions $\omega(x, y, t, \mu)$ of equation (1) are obviously unstable if the operator $S(\mu_0) : W_2^2 \to W_2^2$ has at least one eigenvalue with a positive real part. Therefore (due to Lemma 1), a natural necessary condition for these solutions to be stable is the requirement that the eigenvalues of the matrices (8) for $\mu = \mu_0$ and all $m \neq m_0$ and $n \neq n_0$ have negative real parts. In this regard, we note that such requirement hold only if $m_0 = n_0 = 0$, i.e. under the assumptions of Theorem 1. Namely, the following theorem is true.

Theorem 3. *Let $m_0^2 + n_0^2 \geqslant 1$. Then the bifurcating solutions $\omega(x, y, t, \mu)$ of equation (1) arising under the assumptions of the Theorem 2, are unstable.*

To prove this theorem, it is sufficient to note that if $m_0^2 + n_0^2 \geqslant 1$ X $\varphi_{m_0n_0}(\mu_0) = \mathrm{tr} B_{m_0n_0}(\mu_0) = 0$, then $\varphi_{00}(\mu_0) = \mathrm{tr} A(\mu_0) > 0$. And this (due to Lemma 1) means that the operator $S(\mu_0) : W_2^2 \to W_2^2$ has an eigenvalue with a positive real part. Thus, the bifurcating solutions $\omega(x, y, t, \mu)$ of equation (1) can be stable only under the conditions of Theorem 1. Lets give he corresponding statement.

For this purpose we note that under the assumptions of Theorem 1, the matrix $A(\mu_0)$ has a pair of purely imaginary eigenvalues $\pm\omega_0 i$, where $\omega_0 = \sqrt{\det A(\mu_0)}$.

Below we assume that the elements $a_{ij}(\mu)$ and $k_{ij}(\mu)$ of matrices $A(\mu)$ and $K(\mu)$ for $\mu = \mu_0$ satisfy the assumption

$$\det A + (m^2 + n^2)^2 \det K > (m^2 + n^2)(a_{11}k_{22} + a_{22}k_{11} - a_{12}k_{21} - a_{21}k_{12}) \tag{12}$$

uniformly in all m and n.

Lemma 2. *Let the assumption (12) hold under the assumptions of theorem 1. Then all different from $\pm\omega_0 i$ eigenvalues of operator $S(\mu_0) : W_2^2 \to W_2^2$ have negative real parts.*

Due to Lemma 1, to prove this statement it is sufficient to show that under its assumptions all eigenvalues of all matrices (8) different from $\pm\omega_0 i$ for $\mu = \mu_0$ have negative real parts. Analysis of characteristic equations of matrices (8) shows that the assumption (12) ensures that this fact hold.

4. ON LYAPUNOV QUANTITIES AND TRANSCRITICITY INDICES

The so-called bifurcation formulas play an important role in the theory of bifurcations of dynamical systems and its applications. This term combines formulas that allow us to obtain numerical characteristics, the knowledge of which allows us to conduct a qualitative analysis of various bifurcation scenarios. The bifurcation formulas can, in particular, include Lyapunov quantities and transcriticity indices. Bifurcation formulas have become particularly popular when studying the Andronov–Hopf bifurcation in finite-dimensional dynamic systems. Here the knowledge of the first Lyapunov quantity L_1 and the transcriticity index γ_1 allows us to determine the direction of bifurcations and the stability of emerging solutions (see, for instance, [1–4]).

Analogs of the first Lyapunov quantity L_1 and the transcriticity index γ_1 for the Andronov–Hopf bifurcation problem of the equation (1) under the assumptions of Theorem 1 are given below.

Under the assumptions of theorem 1, the matrix $A_0 = A(\mu_0)$ has a pair of purely imaginary eigenvalues $\pm\omega_0 i$, where $\omega_0 = \sqrt{\det A(\mu_0)}$. Therefore, there exist nonzero vectors e, g, e^*, $g^* \in R^2$ satisfying equalities

$$A_0(e + ig) = i\omega_0(e + ig), \quad A_0^*(e^* + ig^*) = -i\omega_0(e^* + ig^*); \tag{13}$$

here A_0^* is a transposed matrix. The vectors e, g, e^*, g^* can be considered normalized according to the equalities

$$(e, e^*) = (g, g^*) = 1, \quad (e, g^*) = (g, e^*) = 0. \tag{14}$$

Let

$$e(t) = e\cos 2\pi t - g\sin 2\pi t. \tag{15}$$

Below we also need the following auxiliary designation. Let $y(t)$ be a continuous periodic (period 1) vector function. By y_c and y_s, we denote corresponding $\cos 2\pi t$ and $\sin 2\pi t$ Fourier coefficients of this function.

4.1. The Case of Cubic Nonlinearity

Let us first consider the case when the nonlinearity (2) begins with cubic terms, i.e. let

$$h(w,\mu) = h_3(w,\mu) + \dots \tag{16}$$

Let

$$L_1 = \frac{1}{2}[(y_c, e^*) - (y_s, g^*)]; \tag{17}$$

here $y(t) = h_3(e(t), \mu_0)$.

Theorem 4. *Let, under the assumptions of Theorem 1, the nonlinearity $h(w,\mu)$ be represented as (16). Then the number L_1, defined by equality (17), is the first Lyapunov quantity in the problem on the Andronov–Hopf bifurcation for equation (1).*

In this theorem, the phrase that'... the number L_1 ... is the first Lyapunov quantity... should be understood in the sense that the number L_1 expresses the same properties of the Andronov–Hopf bifurcation for the equation (1), what is the first Lyapunov quantity of the Andronov–Hopf bifurcation in the classical sense, i.e. for differential equations in finite-dimensional phase spaces. This issue is discussed in more detail below.

The validity of the Theorem 4, as well as other theorems given below, can be established according to the same scheme that was used in proving similar statements in [7].

4.2. General Case

Let us now consider the question of calculating the first Lyapunov quantity in the general situation when the nonlinearity (2) begins with quadratic terms. Denote by E_0 a two-dimensional subspace of the space W_2^2 containing the vectors e and g. It is a proper subspace of the operator $S_0 = S(\mu_0)$, corresponding to the simple eigenvalues $\pm i\omega_0$.

The space W_2^2 can be represented as $W_2^2 = E_0 \oplus E^0$, where E^0 is an additional subspace invariant for S_0. The equality $W_2^2 = E_0 \oplus E^0$ defines projectors $P_0 : W_2^2 \to E_0$ and $P^0 : W_2^2 \to E^0$ so that $P^0 = I - P_0$. Since the operator $S_0 : W_2^2 \to W_2^2$ is bounded, the bounded linear operator $B_0 = e^{T_0 S_0} : W_2^2 \to W_2^2$ is defined; here $T_0 = 2\pi/\omega_0$, $\omega_0 = \sqrt{\det A(\mu_0)}$. By construction the operator $I - B_0 + P_0 : W_2^2 \to W_2^2$ invertible.

Next, we define a vector and a matrix

$$\rho_2 = \int_0^1 e^{(1-t)T_0 S_0} h_2(e(t),\mu_0)\,dt, \quad B_2 = \int_0^1 e^{(1-t)T_0 S_0} F_2(t)\,dt, \tag{18}$$

where $e(t)$ is function (15), and $F_2(t)$ defined as $F_2(t) = T_0 h'_{2w}(e(t),\mu_0) e^{T_0 S_0 t}$. Note that by construction, the inclusion $\rho_2 \in E^0$ holds. Finally, let

$$y(t) = g(t) + f_3(t), \tag{19}$$

where

$$g(t) = F_2(t)(I - B_0 + P_0)^{-1}\rho_2, \tag{20}$$

$$f_3(t) = h_3(e(t),\mu_0) + F_2(t)\int_0^t e^{-\tau T_0 S_0} h_2(e(\tau),\mu_0)\,d\tau. \tag{21}$$

Theorem 5. *Under the assumptions of Theorem 1, the first Lyapunov quantity L_1 in the problem on the Andronov–Hopf bifurcation for the equation (1) is defined by the equality*

$$L_1 = \frac{1}{2}[(y_c, e^*) - (y_s, g^*)]; \tag{22}$$

here $y(t)$ is function (19).

4.3. On Transcriticity Indices

Let us assume that under the assumptions of Theorem 1 we have $L_1 \neq 0$. Let

$$\gamma_1 = \varphi'(\mu_0)L_1. \tag{23}$$

Theorem 6. *Let $\gamma_1 > 0$ ($\gamma_1 < 0$). Then existing under the assumptions of Theorem 1 bifurcation solutions $\omega(x, y, t, \mu)$ of equations (1) exist when $\mu < \mu_0$ ($\mu > \mu_0$).*

Theorem 7. *For all small $|\mu - \mu_0|$ bifurcation solutions $\omega(x, y, t, \mu)$ of equation (1) existing under the assumptions of Theorem 1 are asymptotically orbitally stable if $L_1 < 0$; they are unstable if $L_1 > 0$.*

Defined by equation (23) the number γ_1 we call the transcriticity index in the problem of the Andronov–Hopf bifurcation for the equation (1), referring to the fact that the number (23) expresses the same properties of the Andronov–Hopf bifurcation for the equation (1), as the rate of transcriticity index for the Andronov–Hopf bifurcation in the classical sense (see [2, 3, 9]).

5. EXAMPLE

Consider a reaction-diffusion system of the form (see, for instance, [1])

$$\begin{cases} \frac{\partial u}{\partial t} = (\mu - 1)u + a^2 v + k_{11}\Delta u + k_{12}\Delta v + u^2 v, \\ \frac{\partial v}{\partial t} = -\mu u - a^2 v + k_{21}\Delta u + k_{22}\Delta v - u^2 v, \end{cases} \tag{24}$$

where $k_{ij} = k_{ij}(\mu) \geqslant 0$, Δ is the Laplace operator. The system (24) is a system of type (1). We study this system on a rectangle (3) with boundary conditions (4).

The matrix $A(\mu)$ in the equation (24) has the form $A(\mu) = \begin{bmatrix} \mu - 1 & a^2 \\ -\mu & -a^2 \end{bmatrix}$. Since $\varphi(\mu) = \mathrm{tr}A(\mu) = \mu - 1 - a^2$ and $\det A(\mu) = a^2$, when $\mu = \mu_0 = a^2 + 1$ for the equation (24) the assumptions of the Theorem 1 hold. Therefore, the value $\mu = \mu_0 = a^2 + 1$ is the Andronov–Hopf bifurcation point for the equation (24).

Let's define the first Lyapunov quantity L_1 and the transcriticity index γ_1. To do this, taking into account the fact that the nonlinearity in the (24) system is cubic, we use the formulas (17) and (23).

Matrix $A_0 = A(\mu_0)$ has a pair of purely imaginary eigenvalues $\pm ia$. The vectors $e, g, e^*, g^* \in R^2$ satisfying the equalities (13) and (14) for $\omega_0 = a$ can be chosen as

$$e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad g = \begin{bmatrix} -a \\ a \end{bmatrix}, \quad e^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad g^* = \begin{bmatrix} -1/a \\ 0 \end{bmatrix}.$$

The function (15) and the nonlinearity (16) here take the form

$$e(t) = \begin{bmatrix} a \sin 2\pi t \\ \cos 2\pi t - a \sin 2\pi t \end{bmatrix}, \quad h_3(w, \mu) = \begin{bmatrix} u^2 v \\ -u^2 v \end{bmatrix}.$$

Thus

$$h_3(e(t), \mu_0) = \frac{1}{4}(a^2 \cos 2\pi t - 3a^3 \sin 2\pi t - a^2 \cos 6\pi t + a^2 \sin 6\pi t) \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Select the corresponding $\cos 2\pi t$ and $\sin 2\pi t$ Fourier coefficients of this function:

$$y_c = \frac{1}{4}a^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad y_s = -\frac{3}{4}a^3 \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus, according to the formulas (17) and (23) the first Lyapunov quantity and the transcriticity index become

$$L_1 = -\frac{3}{8}a^2, \quad \gamma_1 = -\frac{3}{8}a^2.$$

Based on the Theorems 6 and 7 the following conclusions can be drawn. First, bifurcation solutions $\omega(x, y, t, \mu)$ of equation (24) arise when $\mu > \mu_0$. Second, for all small $|\mu - \mu_0|$, these solutions are asymptotically orbitally stable.

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